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A variational principle for magneto-elastic buckling

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Abstract. A variational principle that can serve as the basis for a magneto-elastic stability (or buckling) problem is constructed. For the two cases of soft ferromagnetic media and superconductors, respectively, it is shown how the variational principle directly yields an explicit expression for the buckling value. The formulation starts from a specific choice for a magneto-elastic Lagrangian L (associated with the so-called Maxwell-Minkowski model for magneto-elastic interactions). For the evaluation of the principle the first and second variations of L are calculated both inside and outside the solid magneto-elastic body. Thus, a general buckling criterion, consisting of an expression for the critical field value, together with a set of constraints for the field variables occurring in the right-hand side of this expression, is constructed. Finally, more detailed formulations are given for, successively, soft ferromagnetic bodies and superconductors. Applications to specific structures, yielding explicit numerical values for the magneto-elastic buckling fields, will be given in a forthcoming paper.

1. Introduction

The last two decades have shown a great progress in the research on magneto-elastic stability problems. Based on the pioneering work of F.C. Moon, several other authors have solved problems in this area of research. For an excellent survey and a very extensive list of references, we refer to the monograph of Moon [1]. Important parts of two IUTAM-symposia in Paris 1983 [2] and in Tokyo 1986 [3] were devoted to related subjects. Mostly, these problems are treated in a classical mechanical way, e.g. by means of establishing a beam or a plate equation, in which the loading terms are of magnetic origin (for a general survey of this method, cf. [4]). An alternative way was followed by Goudjo and Maugin [5] who employed the principle of virtual power for the construction of a stability theory for soft ferromagnetic plates.

In the present paper we shall introduce a variational principle on the basis of which a magneto-elastic stability (or buckling) problem can be formulated in terms of an eigenvalue problem. Explicit formulations for this eigenvalue problem will be given for the two, from a practical point of view, most important cases, i.e. (1) soft ferromagnetic, and (2) superconducting media. For these two cases, we shall show how the variational principle directly yields an explicit expression for the buckling value. The advantage of this method is that, whenever it is possible to determine the solution for the intermediate and perturbed electromagnetic fields, it is just a matter of a simple substitution to obtain the buckling value. However, in complex constructions, as occur in for example fusion reactors and high-field magnetic devices, such exact solutions are not available, and then the variational principle serves as a sound basis for a construction of approximation fields yielding an optimal approximation for the buckling value.

We start this paper by showing in general terms how a magneto-elastic buckling problem can be related to an eigenvalue problem, and how this eigenvalue problem can be formulated as a variational principle (see also [6]). For the formulation of this principle the first and second variation of a so called Lagrangian L is needed. In Section 3 an expression for L is given and the first and second variation of L are evaluated in terms of the perturbed fields (which are perturbations with respect to some intermediate, or pre-buckled, state). In Section 4 we will show that this specific choice for L corresponds to the so called Maxwell-Minkowski model for magneto-elastic interactions (cf. [7]). In Section 5 the general buckling criterion is formulated and the main lines for the procedure to obtain a buckling value are described. Finally, in Sections 6 and 7 more detailed formulations are given for soft ferromagnetic structures and superconductors.

2. General eigenvalue problem

Every equilibrium state of a system of bodies, that is influenced by an external magnetic field in vacuum, is governed by a set of equations and boundary conditions (cf. [1], [7]). Let us denote this set schematically by

$$S_i[\mathbf{B}(\mathbf{x}), \mathbf{M}(\mathbf{x}); T(\mathbf{x}), \mathbf{x}; B_0] = 0, \quad 1 \leqslant i \leqslant N. \tag{2.1}$$

The symbols $\mathbf{B}$, $\mathbf{M}$, T, $\mathbf{x}$ and B_0 refer to the magnetic induction, the magnetization, the stress tensor, the position and the external magnetic field parameter (e.g. the field at infinity), respectively. The symbols S_i enclose various differential operators; some of these operators act on the boundaries of the bodies.

In the theory of stability three equilibrium configurations of the bodies are distinguished, namely the natural or unloaded state, given by (here $\mathbf{x} = \mathbf{X}$)

$$S_{0i}[\mathbf{0}, \mathbf{0}; 0, \mathbf{X}; 0] = 0, \quad 1 \leqslant i \leqslant N, \tag{2.2}$$

the intermediate state, satisfying ($\mathbf{x} = \boldsymbol{\xi}$)

$$S_i^0[\mathbf{B}^0(\boldsymbol{\xi}), \mathbf{M}^0(\boldsymbol{\xi}); T^0(\boldsymbol{\xi}), \boldsymbol{\xi}; B_0] = 0, \quad 1 \leqslant i \leqslant N, \tag{2.3}$$

and the present or spatial state, that differs only slightly from the intermediate state, and is characterized by (2.1) or ($\mathbf{x} = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi})$)

$$S_i[(\mathbf{B}^0 + \mathbf{b})(\boldsymbol{\xi}), (\mathbf{M}^0 + \mathbf{m})(\boldsymbol{\xi}); (T^0 + t)(\boldsymbol{\xi}), \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}); B_0] = 0, \quad 1 \leqslant i \leqslant N. \tag{2.4}$$

The perturbations $\mathbf{b}$, $\mathbf{m}$, t and $\mathbf{u}$ are supposed to be small. Subtraction of (2.3) from (2.4) and neglect of terms of the second order in the perturbations yield a problem that is homogeneous with respect to the perturbations. In the sequel this homogeneous problem is denoted by

$$s_i[\mathbf{b}(\boldsymbol{\xi}), \mathbf{m}(\boldsymbol{\xi}); t(\boldsymbol{\xi}), \mathbf{u}(\boldsymbol{\xi}); B_0] = 0, \quad 1 \leqslant i \leqslant N. \tag{2.5}$$

The symbols s_i refer to linear operators, which contain various differential operators, some of them acting on the intermediate boundaries. For each value of the field parameter B_0 there exists the solution

$$s_i[\mathbf{0}, \mathbf{0}; 0, \mathbf{0}; B_0] = 0, \quad 1 \leqslant i \leqslant N,$$

but we are only interested in those values of B_0 for which

$$(\mathbf{b}, \mathbf{m}; t, \mathbf{u}) \neq (\mathbf{0}, \mathbf{0}; 0, \mathbf{0}), \tag{2.6}$$

is a solution of (2.5). The problem posed by (2.5) and (2.6) is an eigenvalue problem; the perturbations and the field B_0 play the role of the eigenvector and the eigenvalue, respectively. In the theory of stability the eigenvalues are called buckling values. Of course we are especially interested in the lowest buckling value. The eigenvalue problem is linear with respect to the perturbations, but depends on B_0 in a non-linear way.

In many cases neglect of the intermediate deformations is justified; this simplification makes it possible to identify the natural and the intermediate boundaries, thus the intermediate configurations are no longer unknown. However, no simplification of the kind makes the dependence on B_0 of the eigenvalue problem less complicated. Generally it is impossible to solve the buckling value directly from (2.5), (2.6).

The basic idea of a variational principle for magneto-elastic buckling is as follows: Assume that some of the equations (or boundary conditions) (2.1), (2.3), say $1 \leqslant i \leqslant k$, are satisfied a priori,

$$\begin{aligned} S_i[\mathbf{B}(\mathbf{x}), \mathbf{M}(\mathbf{x}); T(\mathbf{x}), \mathbf{x}; B_0] &= 0, \qquad 1 \leqslant i \leqslant k < N, \\ S_i^0[\mathbf{B}^0(\xi), \mathbf{M}^0(\xi); T^0(\xi), \xi; B_0] &= 0, \quad 1 \leqslant i \leqslant k < N, \end{aligned} \tag{2.7}$$

and consider these equations as constraints for the variations of the functionals, the so called Lagrangians,

$$\begin{aligned} L[\mathbf{B}, \mathbf{M}; T; B_0] &= \int_{\mathbb{R}_3} \mathrm{L}[\mathbf{B}(\mathbf{x}), \mathbf{M}(\mathbf{x}); T(\mathbf{x}), \mathbf{x}; B_0]\,\mathrm{dV}, \\ L^0[\mathbf{B}^0, \mathbf{M}^0; T^0; B_0] &= \int_{\mathbb{R}_3^0} \mathrm{L}^0[\mathbf{B}^0(\xi), \mathbf{M}^0(\xi); T^0(\xi), \xi; B_0]\,\mathrm{dV}^0. \end{aligned} \tag{2.8}$$

The integrands of the integrals in the right-hand sides of (2.8), the so called Lagrangian densities, are connected with the sets of equations and boundary conditions (2.1) and (2.3) and need to be specified later on. Evaluation of $S_i - S_i^0$ and $L - L^0$ in terms of the perturbations results in (compare with (2.5))

$$\begin{aligned} s_i[\mathbf{b}(\xi), \mathbf{m}(\xi); t(\xi), \mathbf{u}(\xi); B_0] &= 0, \quad 1 \leqslant i \leqslant k, \\ L - L^0 &= \delta L + J + \mathrm{O}(\varepsilon^3), \end{aligned} \tag{2.9}$$

in which ε denotes the order of magnitude of the perturbations and δL and J are the first and half the second variation of L with respect to the intermediate state. Note that δL

contains only terms of order ε, whereas J contains only terms of order ε^2. If $\mathbf{B}^0$, $\mathbf{M}^0$ and T^0 are chosen in such a way that

$$\delta L = 0 \wedge S_i^0[\mathbf{B}^0(\xi), \mathbf{M}^0(\xi); T^0(\xi), \xi; B_0] = 0, \quad 1 \leqslant i \leqslant k \Leftrightarrow (2.3), \tag{2.10}$$

then it can be proved that

$$\delta J = 0 \wedge s_i[\mathbf{b}(\xi), \mathbf{m}(\xi); t(\xi), \mathbf{u}(\xi); B_0] = 0, \quad 1 \leqslant i \leqslant k \Leftrightarrow (2.5). \tag{2.11}$$

Here, δJ is the first variation of J which is defined as

$$\delta J(\mathbf{b}, \mathbf{m}; t, \mathbf{u} \,|\, \mathbf{b}_1, \mathbf{m}_1; t_1, \mathbf{u}_1 \,|\, B_0)$$

$$= \varepsilon \lim_{\varepsilon_1 \downarrow 0} \frac{1}{\varepsilon_1} [J(\mathbf{b} + \varepsilon_1 \mathbf{b}_1, \mathbf{m} + \varepsilon_1 \mathbf{m}_1; t + \varepsilon_1 t_1, \mathbf{u} + \varepsilon_1 \mathbf{u}_1; B_0) - J(\mathbf{b}, \mathbf{m}; t, \mathbf{u}; B_0)]. \tag{2.12}$$

Hence, the eigenvalue problem (2.5), (2.6) is equivalent to

$$\begin{aligned} &\delta J = 0, (\mathbf{b}, \mathbf{m}; t, \mathbf{u}) \neq (\mathbf{0}, \mathbf{0}; 0, \mathbf{0}), \\ &s_i[\mathbf{b}(\xi), \mathbf{m}(\xi); t(\xi), \mathbf{u}(\xi); B_0] = 0, \quad 1 \leqslant i \leqslant k. \end{aligned} \tag{2.13}$$

Here, it is assumed that the intermediate fields are already known.

From the fact that J is a homogeneous and quadratic functional with regard to the perturbations one can deduce the important property

$$\delta J = 0 \Rightarrow J = 0. \tag{2.14}$$

The equivalence of the eigenvalue problem to (2.13) and the property (2.14) imply that any reasonable approximation for the perturbations leads us to a good approximation for the buckling value B_0 (see also Section 5).

3. Statement and evaluation of the Lagrangian

In this section we shall postulate an explicit expression for the Lagrangian in terms of the magnetic field in and outside the deformed body. As indicated in the preceding section these fields are considered as perturbations with respect to some intermediate state. By an evaluation of the Lagrangian with respect to the perturbations, an explicit representation for formula $(2.9)^2$ will be obtained.

We only consider static situations in which one single, simply connected body is influenced by a uniform field $\mathbf{B}_0$. The body is assumed to be magnetizable and non-conducting. For this case, a specific expression for the Lagrangian density L is postulated (note that in a static version $-$L is equal to the energy density). This choice is justified by the fact that a variation of the Lagrangian, under the proper constraints, yields a set of equations and boundary conditions, known in literature as the Maxwell-Minkowski model (cf. [7]). Our choice of L is based upon the form of the electromagnetic energy density for the Maxwell-Minkowski model as given in [7], page 55, i.e. (for $\mathbf{E} = \mathbf{0}$)

$$\tfrac{1}{2}\mu_0(\mathbf{H}, \mathbf{H}).$$

We note that other forms for the energy density are possible and admissible. For instance, the choice of (see [7], page 72)

$$\frac{1}{2\mu_0}(\mathbf{B}, \mathbf{B})$$

would result in the so called Amperean-Current model as is used most frequently by e.g. F. Moon [1]. Hutter and van de Ven showed in [7] that these models are completely equivalent.

The present configuration of the body, its boundary and the vacuum are denoted by G^-, ∂G and G^+, respectively. Then, the Lagrangian density is chosen as (an upper index $^+$ stands for a value outside the body and $^-$ for a value inside the body)

$$\mathrm{L}^+ = -\tfrac{1}{2}\mu_0(\mathbf{H}^+, \mathbf{H}^+) + \frac{1}{2}\frac{1}{\mu_0}B_0^2, \quad \mathrm{L}^- = -\tfrac{1}{2}\mu_0(\mathbf{H}^-, \mathbf{H}^-) + \frac{1}{2}\frac{1}{\mu_0}B_0^2 - \varrho U, \tag{3.1}$$

accompanied by the constraints

$$\begin{aligned}
&\mathbf{B}^+ = \operatorname{curl}\mathbf{A}^+, \quad \mathbf{M}^+ = \mathbf{0}, \quad \mathbf{x} \in G^+;\\
&\mathbf{A}^+ = \mathbf{A}^-, \quad \mathbf{x} \in \partial G;\\
&\mathbf{B}^- = \operatorname{curl}\mathbf{A}^-, \quad T = \varrho\frac{\partial U}{\partial F}F^T, \quad \varrho J_F = \varrho_0, \quad \mathbf{x} \in G^-;\\
&\mathbf{B}^+ \to \mathbf{B}_0, \quad |\mathbf{x}| \to \infty;
\end{aligned} \tag{3.2}$$

where ϱ and ϱ_0 are the mass densities in the present and the natural state, respectively, and $\mathbf{H}$, F and J_F are the magnetic field, the deformation gradient and the Jacobian defined by successively

$$\mathbf{H}^+ = \frac{1}{\mu_0}\mathbf{B}^+, \quad \mathbf{H}^- = \frac{1}{\mu_0}\mathbf{B}^- - \varrho\mathbf{M}^-, \quad F = \frac{\partial\mathbf{x}}{\partial\mathbf{X}}, \quad J_F = \det F, \tag{3.3}$$

where μ_0 is the magnetic permeability in vacuum. Furthermore, the function $U = U(F, \mathbf{M})$ is the internal energy density. Finally $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is some vector potential introduced in order to assure that $\mathbf{B}$ satisfies

$$\begin{aligned} \operatorname{div} \mathbf{B}^{\pm} &= 0, \qquad \mathbf{x} \in G^{\pm}; \\ (\mathbf{B}^{+}, \mathbf{n}) &= (\mathbf{B}^{-}, \mathbf{n}), \quad \mathbf{x} \in \partial G, \end{aligned} \tag{3.4}$$

where $\mathbf{n}$ is the unit normal on ∂G.

NOTE: Requirements of objectivity imply that U can only be a function of tensorial variables which are invariable with respect to observer transformations. This condition can be satisfied by taking

$$U = U(E, \mathbf{\Lambda}), \tag{3.5}$$

where E is the Lagrangian deformation tensor and $\mathbf{\Lambda}$ is the invariable magnetization given by

$$E = \tfrac{1}{2}(F^T F - I), \mathbf{\Lambda} = F^T \mathbf{M}, \tag{3.6}$$

respectively. For the moment however, there is no need for this somewhat more complex formulation, but we shall return to this later on.

General evaluation procedure

For the derivation of expressions for δL and $J = \delta^2 L/2$ an expansion of $L - L^0$ in terms of ε up to and including terms of order ε^2 is needed. For this purpose, a more precise definition of the perturbations is required, as will be given below.

The relation between the Euler coordinates $\mathbf{x}$ and $\boldsymbol{\xi}$ of the perturbed (or present) and intermediate state, respectively, and the displacement $\mathbf{u}$ is

$$\mathbf{x} = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}). \tag{3.7}$$

Inside the body we prefer a formulation in terms of the coordinate $\boldsymbol{\xi}$ and, therefore, we define

$$\mathbf{B}^{-}(\mathbf{x}) = \hat{\mathbf{B}}^{-}(\boldsymbol{\xi}) = \mathbf{B}^{0-}(\boldsymbol{\xi}) + \mathbf{b}^{-}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in G^{0-}, \tag{3.8}$$

with analogous definitions for $\mathbf{h}^{-}(\boldsymbol{\xi})$, $\mathbf{m}^{-}(\boldsymbol{\xi})$, $\mathbf{a}^{-}(\boldsymbol{\xi})$ and $t(\boldsymbol{\xi})$. In the vacuum, material coordinates are meaningless and, so, we are bounded to a formulation of the vacuum perturbations in terms of the local coordinate $\mathbf{x}$ and therefore, we define

$$\mathbf{B}^{+}(\mathbf{x}) = \mathbf{B}^{0+}(\mathbf{x}) + \mathbf{b}^{+}(\mathbf{x}), \quad \mathbf{x} \in G^{+}, \tag{3.9}$$

with analogous definitions for $\mathbf{h}^{+}(\mathbf{x})$, $\mathbf{m}^{+}(\mathbf{x})$ and $\mathbf{a}^{+}(\mathbf{x})$. The use of different coordinates in G^{-} and G^{+} will give rise to some extra terms in the linearized boundary conditions as we shall see later on (e.g. (3.15), see also [8], (3.13), (3.14)).

The order of magnitude of the perturbations, ε, is expressed by

$$\varepsilon = |\partial \mathbf{u}/\partial \boldsymbol{\xi}| \ll 1, \tag{3.10}$$

and it is supposed that

$$|\mathbf{b}| = O(\varepsilon|\mathbf{B}^0|), \quad |\mathbf{h}| = O(\varepsilon|\mathbf{H}^0|), \text{ etc.} \tag{3.11}$$

In the sequel most of the relations will be written in the usual tensor notation, and δ_{ij} and e_{ijk} will be the Kronecker delta and the alternating tensor, respectively. Moreover, differentiation with respect to a coordinate is denoted by a (lower case) letter preceded by a comma, but we have to distinguish between differentiation in G^- and G^+. This means that one has to read $_{,i}$ as

$$_{,i} = \partial/\partial\xi_i, \quad \boldsymbol{\xi} \in G^{0-}, \quad i = 1, 2, 3,$$

$$_{,i} = \partial/\partial x_i, \quad \mathbf{x} \in G^+, \quad i = 1, 2, 3. \tag{3.12}$$

In order to obtain the linearized constraints (2.9)[1], a linearization of the equations and boundary conditions (3.2) is required. Since this linearization is straightforward we give at once the results:

$$\begin{aligned}
&b_i^+ = e_{ijk}a_{k,j}^+, \quad m_i^+ = 0, && \mathbf{x} \in G^{0+};\\
&b_i^- = e_{ijk}(a_{k,j}^- - A_{k,l}^{0-}u_{l,j}), \quad \varrho = \varrho^0(1 - u_{k,k}), &&\\
&t_{ij} = -T_{ij}^0 u_{k,k} + T_{ik}^0 u_{j,k} + \varrho^0(c_{ikjl}^{u0}u_{k,l} + c_{ikj}^{um0}m_k^-), && \boldsymbol{\xi} \in G^{0-};\\
&a_i^+ - a_i^- = -A_{i,j}^{0+}u_j, && \boldsymbol{\xi} \in \partial G^0;\\
&b_i^+ \to 0, && |\mathbf{x}| \to \infty,
\end{aligned} \tag{3.13}$$

in which the material coefficients c_{ijkl}^{u0} and c_{ijk}^{um0} are defined by

$$c_{ijkl}^{u0} = \left[\frac{\partial^2 U}{\partial F_{i\alpha}\partial F_{j\beta}}\right]^0 F_{k\alpha}^0 F_{l\beta}^0, \; c_{ijk}^{um0} = \left[\frac{\partial^2 U}{\partial F_{i\alpha}\partial M_j}\right]^0 F_{k\alpha}^0. \tag{3.14}$$

We note that in the derivation of the linearized boundary condition (3.13)[6] the following result is used (we at once give a second-order expression because this is needed in the sequel):

Let $\mathbf{x} \in \partial G$ and $\boldsymbol{\xi} \in \partial G^0$ be material points of the boundary, then

$$\begin{aligned}
A_i^+(\mathbf{x}) - A_i^-(\mathbf{x}) &= (A_i^{0+} + a_i^+)(\mathbf{x}) - (A_i^{0-} + a_i^-)(\boldsymbol{\xi})\\
&= (A_i^{0+} + a_i^+)(\boldsymbol{\xi} + \mathbf{u}) - (A_i^{0-} + a_i^-)(\boldsymbol{\xi})\\
&= [A_i^{0+}(\boldsymbol{\xi}) - A_i^{0-}(\boldsymbol{\xi})] + [a_i^+(\boldsymbol{\xi}) - a_i^-(\boldsymbol{\xi}) + A_{i,j}^{0+}(\boldsymbol{\xi})u_j(\boldsymbol{\xi})\\
&\quad + a_{i,j}^+u_j + \tfrac{1}{2}A_{i,jk}^{0+}u_ju_k] + O(\varepsilon^3), \quad \boldsymbol{\xi} \in \partial G^0.
\end{aligned} \tag{3.15}$$

Besides the linear constraints (3.13) we need, for an explicit formulation of our variational principle, an expression for the second-order functional J, as introduced in $(2.9)^2$. The derivation of the expression for J requires the second-order approximations of $\mathbf{B}^\pm$ and $\mathbf{H}^\pm$ as can be derived from $(3.2)^{1,4}$ and $(3.3)^{1,2}$, respectively. Using

$$\frac{\partial}{\partial x_i} = \frac{\partial \xi_j}{\partial x_i}\frac{\partial}{\partial \xi_j} = \left[\delta_{ij} - \frac{\partial u_j}{\partial x_i}\right]\frac{\partial}{\partial \xi_j} = [\delta_{ij} - u_{j,i} + u_{j,k}u_{k,i} + O(\varepsilon^3)]\frac{\partial}{\partial \xi_j}, \quad \boldsymbol{\xi} \in G^{0-}, \tag{3.16}$$

and

$$\frac{\varrho}{\varrho_0} = 1 - u_{k,k} + \tfrac{1}{2}(u_{k,k}u_{l,l} + u_{k,l}u_{l,k}) + O(\varepsilon^3), \quad \boldsymbol{\xi} \in G^{0-}, \tag{3.17}$$

we obtain

$$\begin{aligned} b_i^+ &= e_{ijk}a_{k,j}^+, \quad \mathbf{x} \in G^{0+}; \\ b_i^- &= [e_{ijk}(a_{k,j}^- - A_{k,l}^{0-}u_{l,j})] + [e_{ijk}(A_{k,m}^{0-}u_{m,l} - a_{k,l}^-)u_{l,j}], \\ \mu_0 h_i^- &= [b_i^- - \mu_0\varrho^0(m_i^- - u_{k,k}M_i^{0-})] \\ &\quad + [\mu_0\varrho^0(u_{k,k}m_i^- - \tfrac{1}{2}(u_{k,k}u_{l,l} + u_{k,l}u_{l,k})M_i^{0-})], \quad \boldsymbol{\xi} \in G^{0-}. \end{aligned} \tag{3.18}$$

By substitution of

$$A_{k,l}^0 = A_{l,k}^0 - e_{klm}B_m^0, \tag{3.19}$$

which is equivalent to $(3.2)^{1,4}$, into $(3.18)^2$, an alternative expression for $\mathbf{b}^-$ is obtained, which is more convenient for our later elaborations. The result is

$$\begin{aligned} b_i^- &= [e_{ijk}(a_k^- - A_{l,k}^{0-}u_l)_{,j} + u_{i,j}B_j^{0-} - u_{k,k}B_i^{0-}] \\ &\quad + [e_{ijk}(A_{k,m}^{0-}u_{m,l} - a_{k,l}^-)u_{l,j}] + O(\varepsilon^3), \quad \boldsymbol{\xi} \in G^{0-}. \end{aligned} \tag{3.20}$$

Substitution of (3.20) into $(3.18)^3$ yields

$$\begin{aligned} \mu_0 h_i^- &= [e_{ijk}(a_k^- - A_{l,k}^{0-}u_l)_{,j} + u_{i,j}B_j^{0-} - u_{k,k}\mu_0 H_i^{0-} - \mu_0\varrho^0 m_i^-] \\ &\quad + [e_{ijk}(A_{k,m}^{0-}u_{m,l} - a_{k,l}^-)u_{l,j} + \mu_0\varrho^0(u_{k,k}m_i^- \\ &\quad - \mu_0\varrho^0\tfrac{1}{2}(u_{k,k}u_{l,l} + u_{k,l}u_{l,k})M_i^{0-})] + O(\varepsilon^3), \quad \boldsymbol{\xi} \in G^{0-}. \end{aligned} \tag{3.21}$$

The relations $(3.18)^1$, (3.20) and (3.21) will now be used for the determination of J. This will be done in two steps and we start with the material part.

Evaluation of the material Lagrangian L^-

By virtue of the mass balance there exists a relation between the material volume elements dV and dV^0 in the perturbed and intermediate state, respectively, which enables us to transform L^- into an integral with domain G^{0-}. With L^- according to $(3.1)^2$ we then find

$$L^- - L^{0-} = \int_{G^-} L^- \, dV - \int_{G^{0-}} L^{0-} \, dV^0 = \int_{G^{0-}} \left(\frac{\varrho^0}{\varrho} L^- - L^{0-} \right) dV^0$$

$$= \int_{G^{0-}} \left[-\varrho^0 (U - U^0) - \frac{\varrho^0}{\varrho} \frac{1}{2} \mu_0 ((\mathbf{H}, \mathbf{H}) - (\mathbf{H}^0, \mathbf{H}^0)) - \left(\frac{\varrho^0}{\varrho} - 1 \right) \frac{1}{2} \mu_0 (\mathbf{H}^0, \mathbf{H}^0)) + \left(\frac{\varrho^0}{\varrho} - 1 \right) \frac{1}{2\mu_0} B_0^2 \right]^- dV^0. \tag{3.22}$$

Since the present part only concerns fields inside the body, there is no difficulty in a temporary omission of the upper index$^-$. Using the reciprocal relation of (3.17) for the mass density and (3.21), we obtain

$$- \frac{\varrho^0}{\varrho} \frac{1}{2} \mu_0 [(\mathbf{H}, \mathbf{H}) - (\mathbf{H}^0, \mathbf{H}^0)] - \left(\frac{\varrho^0}{\varrho} - 1 \right) \frac{1}{2} \mu_0 (\mathbf{H}^0, \mathbf{H}^0) = - \frac{\varrho^0}{\varrho} \frac{1}{2} \mu_0 [2(\mathbf{H}^0, \mathbf{h}) + (\mathbf{h}, \mathbf{h})]$$

$$- \left(\frac{\varrho^0}{\varrho} - 1 \right) \frac{1}{2} \mu_0 (\mathbf{H}^0, \mathbf{H}^0) = [-e_{ijk}(a_k - A^0_{l,k} u_l)_{,j} H^0_i - T^{M0}_{ij} u_{i,j} + \mu_0 \varrho^0 m_i H^0_i]$$

$$+ [-\tfrac{1}{2}\mu_0 h_i h_i - u_{k,k} \mu_0 h_i H^0_i - \tfrac{1}{4}\mu_0 (u_{k,k} u_{l,l} - u_{k,l} u_{l,k}) H^0_i H^0_i - e_{ijk}(A^0_{k,m} u_{m,l} - a_{k,l}) u_{l,j} H^0_i$$

$$- \mu_0 \varrho^0 u_{k,k} m_i H^0_i + \mu_0 \varrho^0 \tfrac{1}{2} (u_{k,k} u_{l,l} + u_{k,l} u_{l,k}) M^0_i H^0_i] + O(\varepsilon^3), \quad \boldsymbol{\xi} \in G^{0-}, \tag{3.23}$$

in which T^M_{ij} is the so called Maxwell stress tensor defined by

$$T^M_{ij} = H_i B_j - \tfrac{1}{2} \mu_0 H_k H_k \delta_{ij}. \tag{3.24}$$

A Taylor expansion of $\varrho^0(U - U^0)$ in terms of derivatives with respect to the intermediate state yields

$$\varrho^0 (U - U^0) = \left[\varrho^0 \left(\frac{\partial U}{\partial F_{i\alpha}} \right)^0 (F_{i\alpha} - F^0_{i\alpha}) + \varrho^0 \left(\frac{\partial U}{\partial M_i} \right)^0 m_i \right]$$

$$+ \tfrac{1}{2} \varrho^0 \left[\left(\frac{\partial^2 U}{\partial F_{i\alpha} \partial F_{j\beta}} \right)^0 (F_{i\alpha} - F^0_{i\alpha})(F_{j\beta} - F^0_{j\beta}) + 2 \left(\frac{\partial^2 U}{\partial F_{i\alpha} \partial M_j} \right)^0 (F_{i\alpha} - F^0_{i\alpha}) m_j \right.$$

$$\left. + \left(\frac{\partial^2 U}{\partial M_i \partial M_j} \right)^0 m_i m_j \right] + O(\varepsilon^3), \quad \boldsymbol{\xi} \in G^{0-},$$

or

$$\varrho^0(U - U^0) = \left[T^0_{ij}u_{i,j} + \varrho^0\left(\frac{\partial U}{\partial M_i}\right)^0 m_i\right]$$

$$+ \tfrac{1}{2}\varrho^0[c^{u0}_{ijkl}u_{i,k}u_{j,l} + 2c^{um0}_{ijk}u_{i,k}m_j + c^{m0}_{ij}m_im_j] + O(\varepsilon^3), \quad \xi \in G^{0-}, \tag{3.25}$$

where T^0_{ij} is the intermediate stress tensor (cf. (3.2)), c^{u0}_{ijkl} and c^{um0}_{ijk} are the material coefficients as defined by (3.14) and

$$c^{m0}_{ij} = \left(\frac{\partial^2 U}{\partial M_i \partial M_j}\right)^0. \tag{3.26}$$

Substitution of (3.25), (3.23) and (3.17) into (3.22) results in a formulation of $(L^- - L^{0-})$ in terms of the independent perturbations $\mathbf{a}^-$, $\mathbf{m}^-$ and $\mathbf{u}$. Decomposing this result into a term δL^- that only contains terms of order ε and a term J^- of order ε^2, we obtain

$$L^- - L^{0-} = \delta L^- + J^- + O(\varepsilon^3), \tag{3.27.1}$$

where

$$\delta L^- = \int_{G^{0-}}\left[-e_{ijk}(a_k - A^0_{l,k}u_l)_{,j}H^0_i - (T^0_{ij} + T^{M0}_{ij})u_{i,j}\right.$$
$$\left. + \varrho^0 m_i\left(\mu_0 H^0_i - \left(\frac{\partial U}{\partial M_i}\right)^0\right) + \frac{1}{2\mu_0}B_0^2 u_{k,k}\right]^- \mathrm{d}V^0 \tag{3.27.2}$$

and

$$J^- = \int_{G^{0-}}\left[-\tfrac{1}{2}\varrho^0(c^{u0}_{ijkl}u_{i,k}u_{j,l} + 2c^{um0}_{ijk}u_{i,k}m_j + c^{m0}_{ij}m_im_j) - \tfrac{1}{2}\mu_0 h_i h_i\right.$$
$$- u_{k,k}\mu_0(h_i + \varrho^0 m_i)H^0_i - \tfrac{1}{4}\mu_0(u_{k,k}u_{l,l} - u_{k,l}u_{l,k})H^0_iH^0_i$$
$$+ \tfrac{1}{2}\mu_0\varrho^0 M^0_i H^0_i(u_{k,k}u_{l,l} + u_{k,l}u_{l,k}) + e_{ijk}(a_{k,l} - A^0_{k,m}u_{m,l})u_{l,j}H^0_i$$
$$\left. + \frac{1}{4\mu_0}B_0^2(u_{l,l}u_k - u_{k,l}u_l)_{,k}\right]^- \mathrm{d}V^0. \tag{3.27.3}$$

By means of Gauss's divergence theorem an alternative formula for δL^- can be deduced, in which derivatives of the perturbations $\mathbf{a}$, $\mathbf{m}$ and $\mathbf{u}$ are absent, namely

$$\delta L^- = \int_{G^{0-}}\left[-e_{ijk}H^0_{k,j}a_i + [(T^0_{ij} + T^{M0}_{ij})_{,j} + A^0_{i,j}e_{jkl}H^0_{l,k}]u_i\right.$$
$$\left. + \left(\mu_0 H^0_i - \left(\frac{\partial U}{\partial M_i}\right)^0\right)\varrho^0 m_i\right]^- \mathrm{d}V^0 + \int_{\partial G^0}\left[e_{ijk}H^0_k N^0_j a_i\right.$$
$$\left. + \left[-(T^0_{ij} + T^{M0}_{ij})N^0_j - A^0_{i,j}e_{jkl}H^0_l N^0_k + \frac{1}{2\mu_0}B_0^2 N^0_i\right]u_i\right]^- \mathrm{d}V^0, \tag{3.28}$$

where $\mathbf{N}^0$ is the unit normal on ∂G^0.

As the second step in the procedure for the determination of J, we proceed with the vacuum part L^+ of the Lagrangian.

Evaluation of the vacuum Lagrangian L^+

In terms of the Lagrangian densities L^+ and L^{0+} the variation $(L^+ - L^{0+})$ is defined by

$$L^+ - L^{0+} = \int_{G^+} \mathrm{L}^+(\mathbf{x})\, \mathrm{dV} - \int_{G^{0+}} \mathrm{L}^{0+}(\mathbf{x})\, \mathrm{dV}^0, \tag{3.29}$$

i.e as integrals over infinite domains G^+ and G^{0+}, respectively. The behaviour of the magnetic induction $\mathbf{B}^+$ at infinity according to $(3.2)^7$, however, guarantees the existence of L^+ and L^{0+} (that is to say the above integral expressions for L^+ and L^{0+} are convergent).

Firstly, we shall transform the integral for L^+ into one over the intermediate configuration G^{0+}. To this end, we introduce two auxiliary vector functions $\mathbf{W}(\mathbf{x})$ and $\mathbf{W}^0(\mathbf{x})$ by

$$\begin{aligned} \mathrm{L}^+(\mathbf{x}) &= \operatorname{div} \mathbf{W}(\mathbf{x}), \quad \mathbf{x} \in G^+, \\ \mathrm{L}^{0+}(\mathbf{x}) &= \operatorname{div} \mathbf{W}^0(\mathbf{x}), \quad \mathbf{x} \in G^{0+}. \end{aligned} \tag{3.30}$$

The existence of such functions is ensured but they are not determined by (3.30) at all (if $\mathbf{W}(\mathbf{x})$ satisfies (3.30) then $\mathbf{W}(\mathbf{x}) + \operatorname{curl} \mathbf{V}(\mathbf{x})$ also satisfies (3.30)). But, as the auxiliary functions will not occur in the final formula for $(L^+ - L^{0+})$, this indeterminacy is totally irrelevant. Using (3.30) and Gauss's divergence theorem we derive straightforwardly

$$\begin{aligned} L^+ - L^{0+} &= \int_{G^+} \operatorname{div} \mathbf{W}(\mathbf{x})\, \mathrm{dV} - \int_{G^{0+}} \operatorname{div} \mathbf{W}^0(\mathbf{x})\, \mathrm{dV}^0 \\ &= -\int_{\partial G} (\mathbf{W}(\mathbf{x}), \mathbf{n})\, \mathrm{dS} + \int_{\partial G^0} (\mathbf{W}^0(\boldsymbol{\xi}), \mathbf{N}^0)\, \mathrm{dS}^0, \end{aligned} \tag{3.31}$$

in which dS and dS^0 denote the surface elements on ∂G and ∂G^0, respectively. The connection between the directed surface elements is (cf. [9], Eq. (21), page 61)

$$\mathbf{n}\, \mathrm{dS} = \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right)\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right)^T \mathbf{N}^0\, \mathrm{dS}^0 = \frac{\varrho^0}{\varrho}\left(I - \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T \mathbf{N}^0\, \mathrm{dS}^0, \tag{3.32}$$

in which I is the unity tensor. This relation is used in (3.31) to transform the integral over ∂G into an integral over ∂G^0, resulting in

$$L^+ - L^{0+} = \int_{\partial G^0} \left(\mathbf{W}^0(\boldsymbol{\xi}) - \frac{\varrho^0}{\varrho}\left(I - \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) \mathbf{W}(\boldsymbol{\xi} + \mathbf{u}), \mathbf{N}^0\right) \mathrm{dS}^0. \tag{3.33}$$

With

$$\mathbf{W}(\mathbf{x}) = \mathbf{W}^0(\mathbf{x}) + \mathbf{w}(\mathbf{x}), \quad |\mathbf{w}| = O(\varepsilon|\mathbf{W}^0|), \tag{3.34}$$

the integrand of the integral in the right hand side of (3.33) can be evaluated in terms of ε yielding

$$\{W_i^0 - [1 + u_{k,k} + \tfrac{1}{2}(u_{k,k}u_{l,l} - u_{k,l}u_{l,k})]\cdot[\delta_{ij} - u_{i,j} + u_{i,k}u_{k,j}]$$

$$\cdot [W_j^0 + W_{j,k}^0 u_k + \tfrac{1}{2}W_{j,kl}^0 u_k u_l + w_j + w_{j,k}u_k]\}N_i^0 + O(\varepsilon^3) = \{[u_{i,j}W_j^0 - W_{i,j}^0 u_j - u_{j,j}W_i^0]$$

$$-w_i + [u_{i,j}w_j - w_{i,j}u_j - u_{j,j}w_i] + \tfrac{1}{2}[u_{i,j}(W_{j,k}^0 u_k - u_{j,k}W_k^0) - (W_{i,k}^0 u_k - u_{i,k}W_k^0)_{,j}u_j$$

$$+(W_{j,k}^0 u_k - u_{j,k}W_k^0)_{,j}u_i - u_{j,j}(W_{i,k}^0 u_k - u_{i,k}W_k^0)] + \tfrac{1}{2}[(u_{k,k}u_i - u_{i,k}u_k)_{,j}W_j^0$$

$$- W_{i,j}^0(u_{k,k}u_j - u_{j,k}u_k) - (u_{k,k}u_j - u_{j,k}u_k)_{,j}W_i^0] - \tfrac{1}{2}W_{j,jk}^0 u_k u_i\}N_i^0 + O(\varepsilon^3), \tag{3.35}$$

after some rearrangement of terms.

For the further procedure we need the following Lemma, which is a special case of Stokes' theorem.

LEMMA . *If* $\mathbf{f}(\boldsymbol{\xi})$ *and* $\mathbf{g}(\boldsymbol{\xi})$ *possess continuous derivatives in the neighbourhood of* ∂G^0 *then*

$$\int_{\partial G^0} [f_{i,j}g_j - g_{i,j}f_j + f_i g_{j,j} - g_i f_{j,j}]N_i^0 \, dS^0 = 0.$$

Noticing that the integrand can be written as (curl($\mathbf{f} \times \mathbf{g}$), $\mathbf{N}^0$) and taking Stokes' theorem for granted we have a trivial proof of this Lemma.

Substitution of (3.35) into (3.33) and use of the Lemma leads to

$$L^+ - L^{0+} = -\int_{\partial G^0} w_i N_i^0 \, dS^0 - \int_{\partial G^0} W_{j,j}^0 u_i N_i^0 \, dS^0$$

$$- \int_{\partial G^0} [w_{j,j}u_i + \tfrac{1}{2}W_{j,jk}^0 u_i u_k + \tfrac{1}{2}W_{j,j}^0(u_{k,k}u_i - u_{i,k}u_k)]N_i^0 \, dS^0. \tag{3.36}$$

After a transformation of the first integral in (3.36) into a volume integral by means of Gauss' theorem, we eliminate $\mathbf{W}^0$ and $\mathbf{w}$ from (3.36) with the aid of the definitions (3.30). Thus, we arrive at

$$L^+ - L^{0+} = \left\{\int_{G^{0+}} (\mathrm{L}^+ - \mathrm{L}^{0+}) \, dV^0 - \int_{\partial G^0} \mathrm{L}^{0+} u_i N_i^0 \, dS^0\right\} \tag{3.37}$$

$$- \left\{\int_{\partial G^0} [(\mathrm{L}^+ - \mathrm{L}^{0+})u_i + \tfrac{1}{2}\mathrm{L}_{,j}^{0+} u_j u_i + \tfrac{1}{2}\mathrm{L}^{0+}(u_{j,j}u_i - u_{i,j}u_j)]N_i^0 \, dS^0\right\}. \tag{3.38.1}$$

With L^+ according to (3.1) and with (3.18)[1] we have

$$\mathrm{L}^{0+} = -\tfrac{1}{2}\mu_0 H_i^{0+} H_i^{0+} + \frac{1}{2\mu_0} B_0^2, \tag{3.38.1}$$

and

$$\mathrm{L}^+ - \mathrm{L}^{0+} = -\tfrac{1}{2}\mu_0(H_i^{0+} + h_i^+)(H_i^{0+} + h_i^+) - \tfrac{1}{2}\mu_0 H_i^{0+} H_i^{0+}$$

$$= -H_i^{0+} e_{ijk} a_{k,j}^+ - \frac{1}{2\mu_0} b_i^+ b_i^+ . \tag{3.38.2}$$

Substitution of $(3.38)^1$ and $(3.38)^2$ into (3.37) yields

$$L^+ - L^{0+} = -\int_{G^{0+}} e_{ijk} H_i^{0+} a_{k,j}^+ \, dV^0 + \tfrac{1}{2}\mu_0 \int_{\partial G^0} H_k^{0+} H_k^{0+} u_i N_i^0 \, dS^0$$

$$+ \int_{\partial G^0} [e_{jkl} H_j^{0+} a_{l,k}^+ u_i + \tfrac{1}{2}\mu_0 H_k^{0+} H_{k,j}^{0+} u_j u_i + \tfrac{1}{4}\mu_0 H_k^{0+} H_k^{0+} (u_{j,j} u_i - u_{i,j} u_j)] N_i^0 \, dS^0$$

$$- \frac{1}{2\mu_0} \int_{G^{0+}} b_i^+ b_i^+ \, dV^0 - \frac{1}{2\mu_0} B_0^2 \int_{\partial G^0} [u_i + \tfrac{1}{2}(u_{j,j} u_i - u_{i,j} u_j)] N_i^0 \, dS^0. \tag{3.39}$$

Consider the first integral of (3.39), i.e.

$$-\int_{G^{0+}} e_{ijk} H_i^{0+} a_{k,j}^+ \, dV^0 = \int_{G^{0+}} e_{ijk} H_{k,i}^{0+} a_j^+ \, dV^0 + \int_{\partial G^0} e_{ijk} H_i^{0+} a_k^+ N_j^0 \, dS^0$$

$$= \int_{G^{0+}} e_{ijk} H_{k,i}^{0+} a_j^+ \, dV^0 + \int_{\partial G^0} e_{ijk} H_k^{0+} [a_j^- - A_{j,l}^{0+} u_l - a_{j,l}^+ u_l - \tfrac{1}{2} A_{j,lm}^{0+} u_l u_m] N_i^0 \, dS^0, \tag{$*$}$$

where in the second integral $\mathbf{a}^+$ is eliminated in favour of $\mathbf{a}^-$ by means of (3.15). By use of the relation (compare (3.19))

$$A_{i,j}^{0+} = A_{j,i}^{0+} - \mu_0 e_{ijk} H_k^{0+} \tag{3.40}$$

and the definition (see also (3.24))

$$T_{ij}^{M0+} = \mu_0 (H_i^{0+} H_j^{0+} - \tfrac{1}{2} H_k^{0+} H_k^{0+} \delta_{ij}) \tag{3.41}$$

one can derive

$$e_{ijk} H_k^{0+} A_{j,l}^{0+} u_l = e_{ilk} H_k^{0+} A_{j,l}^{0+} u_j - T_{ij}^{M0+} u_j + \tfrac{1}{2}\mu_0 H_k^{0+} H_k^{0+} u_i . \tag{$**$}$$

Substituting $(*)$ and $(**)$ into (3.39) and assembling terms of order ε and those of order ε^2, we ultimately arrive at the following formula for $(L^+ - L^{0+})$ in terms of the independent variables $\mathbf{a}$ (or $\mathbf{b}$) and $\mathbf{u}$ (here δL^+ contains only terms of order ε and J^+ only those of order ε^2 in analogy with (3.27) and (3.28))

$$L^+ - L^{0+} = \delta L^+ + J^+ + O(\varepsilon^3), \tag{3.42.1}$$

where

$$\delta L^{+} = \int_{G^{0+}} e_{ijk} H^{0+}_{k,i} a^{+}_{j} \, dV^{0}$$

$$+ \int_{\partial G^{0}} \left[e_{ijk} H^{0+}_{k} a^{-}_{j} + T^{M0+}_{ij} u_j - A^{0+}_{j,l} e_{lki} H^{0+}_{k} u_j - \frac{1}{2\mu_0} B_0^2 u_i \right] N^{0}_{i} \, dS^{0} \qquad (3.42.2)$$

and

$$J^{+} = -\frac{1}{2\mu_0} \int_{G^{0+}} b^{+}_{i} b^{+}_{i} \, dV^{0} + \int_{\partial G^{0}} \left[H^{0}_{k}(e_{ljk} u_i - e_{ijk} u_l) a_{j,l} + \tfrac{1}{2}\mu_0 H^{0}_{k} H^{0}_{k,j} u_i u_j \right.$$

$$\left. - \tfrac{1}{2} e_{ijm} H^{0}_{m} A^{0}_{j,lk} u_k u_l + \tfrac{1}{4}\left(\mu_0 H^{0}_{k} H^{0}_{k} - \frac{1}{\mu_0} B_0^2 \right)(u_{j,j} u_i - u_{i,j} u_j) \right]^{+} N^{0}_{i} \, dS^{0}. \qquad (3.42.3)$$

The rigorous analytical elaborations presented here enable us to state the final explicit version of the evaluation (2.9)[2]. Adding the formulas (3.27), (3.28) and (3.42) we conclude that

$$L - L^{0} = \delta L + J + O(\varepsilon^{3}), \qquad (3.43.1)$$

in which the first variation δL of L with respect to the intermediate state is given by

$$\delta L = \delta L^{-} + \delta L^{+} = \int_{G^{0-}} \left\{ - e_{ijk} H^{0}_{k,j} a_i + \left[\mu_0 H^{0}_{i} - \left(\frac{\partial U}{\partial M_i} \right)^{0} \right] \varrho^{0} m_i \right.$$

$$\left. + [(T^{0}_{ij} + T^{M0}_{ij})_{,j} + A^{0}_{i,j} e_{jkl} H^{0}_{l,k}] u_i \right\}^{-} dV^{0} + \int_{\partial G^{0}} \{ e_{ijk} (H^{0+}_{k} - H^{0-}_{k}) a^{-}_{j} N^{0}_{i}$$

$$+ [T^{M0+}_{ij} - (T^{M0-}_{ij} + T^{0}_{ij})] u_i N^{0}_{j} + e_{kli} (H^{0+}_{k} A^{0+}_{j,l} - H^{0-}_{k} A^{0-}_{j,l}) u_j N^{0}_{i} \} \, dS^{0}$$

$$- \int_{G^{0+}} e_{ijk} H^{0+}_{k,j} a^{+}_{i} \, dV^{0}, \qquad (3.43.2)$$

whereas the second variation $J = \frac{1}{2}\delta^2 L$ of L reads

$$J = J^{-} + J^{+} = \int_{G^{0-}} \{ -\tfrac{1}{2}\varrho^{0} [c^{u0}_{ijkl} u_{i,k} u_{j,l} + 2 c^{um0}_{ijk} u_{i,k} m_j + c^{m0}_{ij} m_i m_j]$$

$$+ e_{ijk} (a_{k,l} - A^{0}_{k,m} u_{m,l}) H^{0}_{i} u_{l,j} - (B^{0}_{i} H^{0}_{i} - \tfrac{1}{2}\mu_0 H^{0}_{i} H^{0}_{i}) \tfrac{1}{2} (u_{k,k} u_{l,l} - u_{k,l} u_{l,k}) - \tfrac{1}{2}\mu_0 h_i h_i$$

$$- H^{0}_{i} b_i u_{k,k} \}^{-} dV^{0} + \int_{\partial G^{0}} \{ H^{0}_{k} (e_{ljk} u_i - e_{ijk} u_l) a_{j,l} + \tfrac{1}{2}\mu_0 H^{0}_{k} H^{0}_{k,j} u_i u_j - \tfrac{1}{2} e_{ijm} H^{0}_{m} A^{0}_{j,kl} u_k u_l$$

$$+ \tfrac{1}{4}\mu_0 H^{0}_{k} H^{0}_{k} (u_{j,j} u_i - u_{i,j} u_j) \}^{+} N^{0}_{i} \, dS^{0} - \frac{1}{2\mu_0} \int_{G^{0+}} b^{+}_{i} b^{+}_{i} \, dV^{0}. \qquad (3.43.3)$$

It should be noted that the terms in L^- and L^+ containing B_0^2 cancel each other. Hence, the term $B_0^2/2\mu_0$ in $L^\pm$ is totally irrelevant to the value of δL or J and was merely added to the Lagrangian density to make the integral L^+ converge.

The formulas (3.43) form the basis for the next sections in which our variational principle is further developed.

4. Consequences of the variations of L and J

Now that we have the disposal of explicit expressions for δL and J we shall show that variation of L and of J results in sets of equations corresponding to the Maxwell-Minkowski model for magneto-elastic interactions. Thus, we have proved that the variational principle described in Section 2 (i.e. (2.10), (2.11)) with the Lagrangian according to Section 3 is equivalent with this model. For the further procedures it is convenient to make some rearrangements in the constraints and the variables. From now one we shall consider $\mathbf{A}^0$, $\mathbf{M}^0$, F^0 and $\mathbf{a}$, $\mathbf{m}$, $\mathbf{u}$ as basic variables and $\mathbf{B}^0$, $\mathbf{H}^0$, ϱ^0, T^0, T^{M0} and $\mathbf{b}$, $\mathbf{h}$ and t as auxiliary variables. We then consider $(3.2)^{1,4,5,6}$, $(3.3)^{1,2}$ and (3.24) as definitions, rather than as constraints, for $\mathbf{B}^{0\pm}$, T^0, ϱ^0, $\mathbf{H}^{0\pm}$ and $T^{M0\pm}$, and in the same sense we consider $(3.13)^{1,3,5}$ and (3.18) (from the latter only the linearized version) as definitions for $\mathbf{b}^\pm$, t and $\mathbf{h}^\pm$. Thus, the only relevant constraints are (see (3.2) and (3.13))

for *the intermediate state*

$$A_i^{0+} = A_i^{0-}, \quad \xi \in \partial G^0; \tag{4.1}$$

$$e_{ijk} A_{k,j}^{0+} \to B_{0i}, \quad |\mathbf{x}| \to \infty,$$

for *the perturbed state*

$$a_i^+ = a_i^- - A_{i,j}^{0+} u_j, \quad \xi \in \partial G^0; \tag{4.2}$$

$$e_{ijk} a_{k,j}^+ \to 0, \quad |\mathbf{x}| \to \infty.$$

We proceed with a more detailed discussion of the procedure described in Section 2. The requirement $\delta L = 0$ applied to $(3.43)^2$ yields

$$e_{ijk} H_{k,j}^{0-} = 0, \quad \mu_0 H_i^{0-} = \left(\frac{\partial U}{\partial M_i}\right)^{0-}, \quad T_{ij,j}^0 + T_{ij,j}^{M0-} = 0, \quad \xi \in G^{0-};$$

$$e_{ijk}(H_k^{0+} - H_k^{0-}) N_j^0 = 0, \quad T_{ij}^0 N_j^0 = (T_{ij}^{M0+} - T_{ij}^{M0-}) N_j^0, \quad \xi \in \partial G^0; \tag{4.3}$$

$$e_{ijk} H_{k,j}^{0+} = 0, \quad \mathbf{x} \in G^{0+}.$$

NOTE: In the derivation of the boundary condition $(4.3)^5$ it is used that $(e_{lki} H_k^0 A_{j,l}^0 N_i^0)$ is continuous across ∂G^0; this is because $\mathbf{A}^0$ (and, hence, also its tangential derivative) and $(\mathbf{N}^0 \times \mathbf{H}^0)$ are continuous across ∂G^0.

Supplementing (4.3) by the constraints (4.1) and the definitions (3.2)[1,4,5,6], (3.3)[1,2] and (3.24), we obtain a set of equations and boundary conditions known as the Maxwell-Minkowski model, here referring to the intermediate state.

The calculation of δJ, the variation of the functional J with respect to the independent perturbations $\mathbf{a}$, $\mathbf{m}$ and $\mathbf{u}$ is similar to the derivation of the expression for J, (3.43)[3]. Therefore, we only state the result:

$$\begin{aligned}
\delta J = &\int_{G^{0-}} \{e_{ijk}(h_k - H^0_{k,l}u_l)_{,j}\delta a_i + [\mu_0 h_i - (c^{um0}_{jik}u_{j,k} + c^{m0}_{ij}m_j)]\varrho^0\delta m_i \\
&+ [(t_{ij} + t^M_{ij})_{,j} - (T^0_{ij} + T^{M0}_{ij})_{,k}u_{k,j} + A^0_{i,j}e_{jkl}(h_l - H^0_{l,m}u_m)_{,k}]\delta u_i\}^- \,\mathrm{dV}^0 \\
&+ \int_{\partial G^0} \{-e_{ijk}[(h^+_k - h^-_k)N^0_j - u_{l,j}N^0_l(H^{0+}_k - H^{0-}_k) + H^{0+}_{k,l}u_l N^0_j]\delta a^-_i \\
&+ [(t^{M+}_{ij} - t^{M-}_{ij} - t_{ij})N^0_j + u_{k,k}(T^{M0+}_{ij} - T^{M0-}_{ij} - T^0_{ij})N^0_j \\
&- u_{j,k}(T^{M0+}_{ik} - T^{M0-}_{ik} - T^0_{ik})N^0_j + A^{0+}_{i,j}e_{jkl}((h^+_l - h^-_l)N^0_k \\
&- u_{m,k}(H^{0+}_l - H^{0-}_l)N^0_m + H^{0+}_{k,m}u_m N^0_k)]\delta u_i\} \,\mathrm{dS}^0 - \int_{G^{0+}} e_{ijk}h^+_{k,j}\delta a^+_i \,\mathrm{dV}^0. \qquad (4.4)
\end{aligned}$$

Here, t^M is the linearization of T^M (since we do not need this later on, we refrain from giving an explicit expression for t^M). From (4.4) we conclude that the requirement $\delta J = 0$ yields the following system of equations and boundary conditions

$$\begin{aligned}
&e_{ijk}(h_k - H^0_{k,l}u_l)^-_{,j} = 0, \quad \mu_0 h^-_i = c^{um0}_{jik}u_{j,k} + c^{m0}_{ij}m^-_j, \\
&(t_{ij} + t^{M-}_{ij})_{,j} - (T^0_{ij} + T^{M0-}_{ij})_{,k}u_{k,j} = 0, \quad \xi \in G^{0-}; \\
&[t^{M+}_{ij} + u_{k,k}T^{M0+}_{ij} - u_{j,k}T^{M0+}_{ik}]N^0_j = [t_{ij} + t^{M-}_{ij} + u_{k,k}(T^0_{ij} + T^{M0-}_{ij}) - u_{j,k}(T^0_{ik} + T^{M0-}_{ik})]N^0_j, \\
&e_{ijk}[(h^+_k - h^-_k)N^0_j - u_{l,j}N^0_l(H^{0+}_k - H^{0-}_k) + H^{0+}_{k,l}u_l N^0_j] = 0, \quad \xi \in \delta G^0; \qquad (4.5) \\
&e_{ijk}h^+_{k,j} = 0, \quad \mathbf{x} \in G^{0+}.
\end{aligned}$$

NOTE: On account of the boundary condition (4.5)[5], which arises from the variation with respect to $\mathbf{a}$, the last term in the coefficient of δu_i in (4.4) vanishes.

Together with the constraints (4.2) and the definitions (3.13)[1,3,5] and (3.18), the set (4.5) amounts to the linearized Maxwell-Minkowski model (cf. [7], section 5.3).

At this stage we have proved the validity of the theory presented in Section 2, that is, we have shown the equivalence between the variational principle (2.10)–(2.11) with L according to (3.1), and the Maxwell-Minkowski model.

5. General buckling criterion

In magneto-elastic stability theory it has been the usual procedure to start from a linearized set of equations for the perturbations, such as e.g. (4.2), (4.5), and to look for a value of the basic field parameter B_0 for which this set has a non-trivial solution. Since an exact 3-dimensional solution for this set is mostly very difficult, one starts looking for adequate approximate solutions. This is usually done in the following way (consult e.g. [1], [8], [10]; see also [6]) which is of special application to slender bodies:

> For a slender body the 3-dimensional displacement **u** is approximated by a 1- or 2-dimensional characteristic displacement parameter w (e.g. a deflection of a central line or plane of the slender body); this w is chosen in such a way that it satisfies the *global* equilibrium equations (i.e. integrated versions of $(4.5)^3$ together with $(4.5)^5$); $\mathbf{h}^{\pm}$ and $\mathbf{m}^-$ are solved from the remaining equations, i.e. $(4.5)^{1,2,4,6}$, in which **u** is replaced by its approximation w; finally, the buckling value is then found as the first eigenvalue for B_0 for which this solution is unequal to the zero-solution.

However, as the "solution" obtained by the procedure described above is not an exact solution of (4.5), but only a reasonable approximation, the calculated value for B_0 is also an approximation.

Let us introduce a scalar η ($0 \leqslant \eta \leqslant 1$) as a measure for the approximation error in the perturbations; then it is evident that, due to the linear character of the perturbed equations, the error in the eigenvalue for B_0 is also of the first order in η. In this respect, the use of our variational principle clearly has an advantage over the method described above. For, in our procedure the error in B_0 is of the second order in η. This can be explained best by first describing the main lines of our method. These lines are successively

i) choose a class of trial functions $\{\mathbf{a}, \mathbf{m}, \mathbf{u}; B_0\}$ satisfying the constraints (4.2);
ii) determine the best member out of this class by setting $\delta_a J = \delta_m J = \delta_u J = 0$;
iii) calculate the buckling value for B_0 from the equation $J = 0$ (see (2.14)).

Due to the stationary behaviour of the quadratic functional J the deviation between the exact buckling value and the approximated one calculated in (iii) is of the order of the square of the deviation between the exact and the approximated perturbations.

The choice of a class of trial functions (point (i)) is usually based on a choice of a displacement field. In practice, buckling theory always applies to slender bodies, such as beams or rods, plates and rings. For slender bodies the displacement in buckling can be characterized by one or two global displacement parameters. Examples of such global displacement parameters are the deflection of the central line of a beam or the normal displacement of the central plane of a thin plate. Here, we always shall approximate the 3-dimensional displacement field **u** by the global displacement parameter pertinent to the type of slender body under consideration. Of course, this global displacement has to satisfy the support conditions of the body. As soon as this choice is made, the constraint $(4.2)^1$ for **a** can be made explicit.

Clearly, it is assumed that the intermediate fields are known (note that these are also needed in the formulation for J, $(3.43)^3$). In principle these fields can be determined from (4.1) and (4.3), but if this is too complicated we have also the disposal of a variational principle for the ξ-state (see $(3.43)^2$). Thus, approximated intermediate fields can be calculated from the variation $\delta L = 0$, if necessary.

In many practical problems, however, the deformations in the ξ-state have only a negligible influence on the buckling value. In these cases, the intermediate state may be replaced by the so called rigid-body state. As long as the shape of the body is not too complicated, the determination of these rigid-body fields is rather simple (at least in comparison with the calculation of the perturbations).

In the next two sections more explicit applications of our variational principle will be given for a) soft ferromagnetic structures, b) superconductors.

6. Soft ferromagnetic structures

A soft ferromagnetic medium is characterized by a linear relationship between the magnetization and the magnetic field. In this section we shall consider soft ferromagnetic media, which, moreover, are isotropic, homogeneous and linearly elastic. Keeping in mind the note at the beginning of Section 3, which states that the internal energy density U must be a function of E and Λ (see (3.5)), we assume U of the form

$$U = \frac{\mathrm{E}}{2\varrho_0(1+\nu)}\left\{\frac{\nu}{1-2\nu}(\mathrm{tr}\ E)^2 + \mathrm{tr}\ (E^2)\right\} + \frac{\varrho_0\mu_0}{2\chi}(\Lambda, \Lambda), \tag{6.1}$$

where E is Young's modulus, ν is Poisson's ratio and χ represents the ferromagnetic susceptibility. The first term in (6.1) is the elastic bending energy and the second term the ferromagnetic energy; magnetostrictive energy is not included in this expression.

In the sequel we suppose that the ferromagnetic susceptibility χ is so large, that χ^{-1} is negligible with respect to unity. As a consequence, all terms containing a factor χ^{-1} will be neglected, which in essence implies that the ferromagnetic term in (6.1) vanishes. The direct consequences of this are that

$$\mathbf{H}^{0-} = \mathbf{h}^- = 0 \tag{6.2}$$

and that (see (3.14) and (3.26))

$$c_{ijk}^{um0} = 0, \quad c_{ij}^{m0} = 0. \tag{6.3}$$

Under these restrictions the system for the intermediate state and the explicit expression for J reduce considerably. From the intermediate state variables only $\mathbf{B}^{0+}$ and, eventually, T^0 are relevant for the rest of this section. Use of (6.1), (6.2) in (4.1), (4.3) and (3.2) yields

$$\begin{aligned}
&B_i^{0+} = e_{ijk}A_{k,j}^{0+}\left(\text{or } B_{i,i}^{0+} = 0, \quad \int_{\partial G^0} B_i^{0+}N_i^0\,\mathrm{d}S^0 = 0\right),\\
&e_{ijk}B_{k,j}^{0+} = 0, \quad \mathbf{x} \in G^{0+};\\
&T_{ij,j}^0 = 0, \quad T_{ij}^0 = \frac{\varrho^0}{\varrho_0}\frac{\mathrm{E}}{1+\nu}F_{i\alpha}^0\left[\frac{\nu}{1-2\nu}E_{\gamma\gamma}^0\delta_{\alpha\beta} + E_{\alpha\beta}^0\right]F_{j\beta}^0, \quad \xi \in G^{0-};\\
&e_{ijk}B_k^{0+}N_j^0 = 0, \quad T_{ij}^0N_j^0 = \frac{1}{2\mu_0}B_j^{0+}B_j^{0+}N_i^0, \quad \xi \in \delta G^0;\\
&B_i^{0+} \to B_{0i}, \quad |\mathbf{x}| \to \infty.
\end{aligned} \tag{6.4}$$

Substitution of (6.1)–(6.3) into (3.43)[3] and elimination of $\mathbf{H}^{0+} = \mathbf{B}^{0+}/\mu_0$ and $\mathbf{A}^{0+}$ in favour of $\mathbf{B}^{0+}$ (by use of (6.4)[1]) and of $\mathbf{a}^+$ in favour of $\mathbf{b}^+$ (by means of the relation $b_i^+ = e_{ijk}a_{k,j}^+$), results in the following simplified expression for J,

$$J = -\frac{1}{2}\int_{G^{0-}}\left[T_{jk}^0 u_{i,k}u_{i,j} + \frac{\varrho^0}{\varrho_0}\frac{\mathrm{E}}{2(1+\nu)}\left(\frac{2\nu}{1-2\nu}B_{ij}^0B_{kl}^0 + B_{ik}^0B_{jl}^0 + B_{il}^0B_{jk}^0\right)u_{i,j}u_{k,l}\right]\mathrm{d}V^0$$

$$+ \frac{1}{\mu_0}\int_{\delta G^0}[B_k^0 b_k u_i + \tfrac{1}{4}(B_k^0B_k^0u_j)_{,j}u_i - \tfrac{1}{4}B_k^0B_k^0u_{i,j}u_j]^+ N_i^0\,\mathrm{d}S^0 - \frac{1}{2\mu_0}\int_{G^{0+}} b_i^+b_i^+\,\mathrm{d}V^0, \tag{6.5}$$

where B_{ij} is the left Cauchy-Green tensor, i.e.

$$B_{ij}^0 = F_{i\alpha}^0F_{j\alpha}^0. \tag{6.6}$$

The intermediate fields are to be calculated from (6.4); the only relevant constraints for the perturbations $\mathbf{b}^+$ and $\mathbf{u}$ are

$$b_i^+ = e_{ijk}a_{k,j}^+, \quad \mathbf{x}\in G^{0+}, \quad \left(\text{or } b_{i,i}^+ = 0, \quad \int_{\partial G^0} b_i^+N_i^0\,\mathrm{d}S^0 = 0\right);$$

$$b_i^+ \to 0, \quad |\mathbf{x}| \to \infty, \tag{6.7}$$

possibly supplemented by some kinematical boundary conditions for $\mathbf{u}$ if the body is supported.

Assuming for a moment that the intermediate fields are known, we have to choose the perturbations $\mathbf{b}^+$ and $\mathbf{u}$ from some admissible class (satisfying the constraints) and to determine the optimal $\mathbf{b}^+$ and $\mathbf{u}$ in this class by variation of J. It is not surprising that, if we choose the perturbations from the complete class of admissible fields, our variation principle will yield an optimal $\mathbf{b}^+$-field that is conservative, i.e. a field that satisfies

$$e_{ijk}b_{k,j}^+ = 0, \quad \mathbf{x}\in G^{0+}. \tag{6.8}$$

For every conservative field $\mathbf{b}^+$ there exists a continuous potential $\psi = \psi(\mathbf{x})$, such that

$$b_i^+ = \psi_{,i}^+, \quad \mathbf{x}\in G^{0+}. \tag{6.9}$$

Motivated by this result, we now choose the perturbation $\mathbf{b}^+$ such that it can be expressed in a scalar field $\psi(\mathbf{x})$ in the way as in (6.9). In order that this is consistent with (6.7)[1], ψ has to satisfy the constraints

$$\Delta\psi = \psi_{,ii} = 0, \quad \mathbf{x}\in G^{0+};$$

$$\psi \to 0, \quad |\mathbf{x}| \to \infty; \tag{6.10}$$

$$\int_{\partial G^0}\frac{\partial\psi}{\partial N^0}\,\mathrm{d}S^0 = 0.$$

Note that ψ is not determined by (6.10), because the value of ψ on the boundary has not yet been specified.

The use of (6.9) enables us to transform the integral over G^{0+} in (6.5) into a surface integral over ∂G^0 by means of Gauss' theorem. With the use of $(6.4)^{2,5}$ we, thus, can write (6.5) in the form

$$J = -\frac{1}{2}\int_{G^{0-}}\left[T^0_{jk}u_{i,k}u_{i,j} + \frac{\varrho^0}{\varrho_0}\frac{E}{2(1+\nu)}\left(\frac{2\nu}{1-2\nu}B^0_{ij}B^0_{kl} + B^0_{ik}B^0_{jl} + B^0_{il}B^0_{jk}\right)u_{i,j}u_{k,l}\right]\mathrm{d}V^0$$
$$+ \frac{1}{2\mu_0}\int_{\partial G^0}\left[(\psi + B^{0+}_k u_k)\frac{\partial\psi}{\partial N^0} + B^{0+}_i u_i\frac{\partial}{\partial N^0}(\psi + B^{0+}_k u_k) - B^{0+}_k u_k B^{0+}_j u_{j,i}N^0_i\right.$$
$$\left. + \tfrac{1}{2}B^{0+}_k B^{0+}_k(u_{j,j}u_i - u_{i,j}u_j)N^0_i\right]\mathrm{d}S^0. \tag{6.11}$$

Variation of J with respect to ψ under the constraints (6.10) results in

$$0 = \delta_\psi J = \frac{1}{2\mu_0}\int_{\partial G^0}\left[\frac{\partial\psi}{\partial N^0}\delta\psi + (\psi + B^{0+}_k u_k)\frac{\partial}{\partial N^0}\delta\psi + B^{0+}_k u_k\frac{\partial}{\partial N^0}\delta\psi\right]\mathrm{d}S^0$$
$$= \frac{1}{\mu_0}\int_{\partial G^0}(\psi + B^{0+}_k u_k)\delta\left(\frac{\partial\psi}{\partial N^0}\right)\mathrm{d}S^0, \tag{6.12}$$

where we have used Green's second identity in the form

$$\int_{\partial G^0}\left[\delta\psi\frac{\partial\psi}{\partial N^0} - \psi\frac{\partial}{\partial N^0}\delta\psi\right]\mathrm{d}S^0 = 0, \tag{6.13}$$

in correspondence with (6.10). Relation (6.12) together with $(6.10)^3$ implies that

$$\psi + B^{0+}_k u_k = \psi_0, \quad \xi \in \partial G^0, \tag{6.14}$$

where ψ_0 is a unique constant. Hence, we conclude that after a (for the moment arbitrary) choice of the field $\mathbf{u}$, that perturbation $\mathbf{b}^+$ that approximates the exact $\mathbf{b}^+$ best is solved from (6.10), (6.14). This result is rather important because in many problems, especially for slender bodies, our knowledge about the form of the displacements is more extensive than that about the perturbed magnetic field. This means that it is easier to make a reasonable choice for $\mathbf{u}$ than for $\mathbf{b}^+$.

In this concept, however, it is necessary to derive from (6.10) and (6.14) by given $\mathbf{u}$ an exact solution for ψ. As long as the shape of the body is not too complicated this can be done (as we shall show in a forthcoming paper), but otherwise a different way must be followed. In the latter case we choose a set of trial functions for ψ out of a class restricted by (6.10) and we determine the optimal ψ by $\delta_\psi J = 0$. Before we can state an ultimate expression for the buckling value, we have one more step to go.

In practice, buckling problems always apply to slender bodies. The buckling problem for a slender body often admits the neglect of the intermediate deformations. In that case we may identify the intermediate state by the undeformed or natural state of the body. Hence $\xi \to \mathbf{X}$ and

$$G^{0\pm} = G_0^{\pm}, \quad \partial G^0 = \partial G_0, \quad \mathbf{N}^0 = \mathbf{N}, \quad B_{ij}^0 = \delta_{ij}, \tag{6.15}$$

by which (6.9) reduces to

$$J = -\frac{1}{2}\int_{G^{0-}}\left[T_{jk}u_{i,k}u_{i,j} + \frac{\mathrm{E}}{1+\nu}\left(\frac{\nu}{1-2\nu}e_{kk}e_{ll} + e_{kl}e_{kl}\right)\right]\mathrm{dV}^0$$
$$+ \frac{1}{2\mu_0}\int_{\partial G^0}\left[(\psi + B_k u_k)\frac{\partial\psi}{\partial N} + B_i u_i\frac{\partial}{\partial N}(\psi + B_k u_k) - B_k u_k B_j u_{j,i} N_i\right.$$
$$\left. + \tfrac{1}{2}B_k B_k(u_{j,j}u_i - u_{i,j}u_j)N_i\right]\mathrm{dS}^0, \tag{6.16}$$

where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{6.17}$$

Here $\mathbf{B}$ ($:=\mathbf{B}^{0+}$) and T ($:=T^0$) are the rigid-body fields which satisfy

$$\begin{aligned} &\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{B} = 0, \quad \mathbf{x}\in G_0^+, \quad \mathbf{X}\in G_0^-; \quad \mathbf{B}\times\mathbf{N} = 0, \quad \mathbf{X}\in\partial G_0; \\ &\int_{\partial G_0}(\mathbf{B}, \mathbf{N})\,\mathrm{dS} = 0; \quad \mathbf{B}\to\mathbf{B}_0, \ |\mathbf{x}|\to\infty \end{aligned} \tag{6.18}$$

and

$$T_{ij,j}^0 = 0, \quad \mathbf{X}\in G_0^-; \quad T_{ij}N_j = \frac{1}{2\mu_0}(\mathbf{B}, \mathbf{B})N_i, \quad \mathbf{X}\in\partial G_0. \tag{6.19}$$

From (6.18) and (6.19) it is evident that the following normalized field quantities ($B_0 = |\mathbf{B}_0|$)

$$\hat{\mathbf{B}} := \mathbf{B}/B_0, \quad \hat{T} := \mu_0 T/B_0^2, \tag{6.20}$$

are independent of B_0 and the same is true for (see (6.10), (6.14))

$$\hat{\psi} := \psi/B_0. \tag{6.21}$$

After having chosen the displacement field $\mathbf{u}$ and the determination of the associated ψ-field (either exactly or by variation), we proceed with the calculation of J according to (6.16).

Then, finally, the buckling value is determined by putting $J = 0$, yielding (with the use of (6.20), (6.21) and omitting the hats)

$$\frac{\mu_0 \mathrm{E}}{B_0^2} = \left\{ \int_{\partial G_0} \left[(\psi + B_k u_k) \frac{\partial \psi}{\partial N} + B_i u_i \frac{\partial}{\partial N} (\psi + B_k u_k) - B_k u_k B_j u_{j,i} N_i \right.\right.$$
$$\left.\left. + \tfrac{1}{2} B_k B_k (u_{j,j} u_i - u_{i,j} u_j) N_i \right] \mathrm{dS}_0 - \int_{G_0^-} T_{jk} u_{i,k} u_{i,j} \, \mathrm{dV}_0 \right\}$$
$$\times \left\{ \frac{1}{1+\nu} \int_{G_0^-} \left[\frac{\nu}{1-2\nu} e_{kk} e_{ll} + e_{kl} e_{kl} \right] \mathrm{dV}_0 \right\}^{-1}. \tag{6.22}$$

In this result, the pre-stresses T still occur. In some cases, e.g. for straight beams, the pre-stresses can be neglected, but a general statement for this is not possible at this stage.

7. Superconductors

The theory of the preceding deals specifically with the case in which a magnetizable body is influenced by an external uniform magnetic field $\mathbf{B}_0$. However, as we shall show in this section, our general variational principle can be equally well applied to superconductors with a prescribed total electric current I_0. In that case the buckling value is the value of I_0, corresponding with the lowest eigenvalue of the general eigenvalue problem of Section 2; here B_0 is replaced by I_0. Since the analysis runs essentially along the same lines, it suffices to point out the main differences and to give only the results.

We consider a superconducting body as a non-magnetizable body, for which the current density $\mathbf{J}$ (per unit of area) is concentrated on the surface of the body, and for which the magnetic field $\mathbf{B}^-$ inside the body vanishes. The current density $\mathbf{J}$ is related to the boundary value of the vacuum field $\mathbf{B}^+$ by

$$\mu_0 \mathbf{J} = \mathbf{n} \times \mathbf{B}^+, \quad \mathbf{x} \in \partial G. \tag{7.1}$$

For reasons of simplicity we only consider one single, simply connected superconductor in a static situation. Bearing in mind that $\mu_0 \mathbf{H}^+ = \mathbf{B}^+, \mathbf{B}^- = \mathbf{B}_0 = 0$ we introduce, in analogy with (3.1) and (3.2), the Lagrangian densities and the constraints as

$$\mathrm{L}^+ = -\frac{1}{2\mu_0} (\mathbf{B}^+, \mathbf{B}^+), \quad \mathrm{L}^- = -\varrho U, \tag{7.2}$$

accompanied by the constraints

$$\begin{aligned} &\mathbf{B}^+ = \operatorname{curl} \mathbf{A}^+, \quad \mathbf{x} \in G^+; \\ &\mathbf{B}^- = 0, \quad T = \varrho \frac{\mathrm{d}U}{\mathrm{d}F} F^{\mathrm{T}}, \quad \varrho J_F = \varrho_0, \quad \mathbf{x} \in G^-; \\ &(\mathbf{n}, \operatorname{curl} \mathbf{A}^+) = 0, \quad (\text{or } \mathbf{A}^+ = \text{constant}), \quad \mathbf{x} \in \partial G; \\ &\mathbf{B}^+ \to \mu_0 I_0 \mathbf{c}(\mathbf{x}), \quad |\mathbf{x}| \to \infty, \end{aligned} \tag{7.3}$$

where the vector potential $\mathbf{A} = \mathbf{A}(\mathbf{x})$ assures that $\mathbf{B}^+$ satisfies (compare with (3.4))

$$\begin{aligned} \operatorname{div} \mathbf{B}^+ &= 0, \quad \mathbf{x} \in G^+; \\ (\mathbf{B}^+, \mathbf{n}) &= 0, \quad \mathbf{x} \in \partial G; \end{aligned} \tag{7.4}$$

and $\mathbf{c}(\mathbf{x})$ is an explicit field, indepenent of the total current I_0, that needs to be specified for the particular case in question. In all cases $\mathbf{c}(\mathbf{x})$ tends to zero at infinity. For a straight, infinitely long conductor,

$$|\mathbf{c}(\mathbf{x})| = \frac{1}{2\pi|\mathbf{x}|}. \tag{7.5}$$

The linearization of the constraints (7.3) is straightforward and the result is

$$\begin{aligned} b_i^+ &= e_{ijk} a_{k,j}^+, \quad \mathbf{x} \in G^{0+}; \\ b_i^- &= 0, \quad t_{ij} = -T_{ij}^0 u_{k,k} + T_{ik}^0 u_{j,k} + \varrho^0 c_{ikjl}^{u0} u_{k,l}, \quad \varrho = \varrho^0(1 - u_{k,k}), \quad \boldsymbol{\xi} \in G^{0-}; \\ a_i^+ &= -A_{i,j}^{0+} u_j, \quad \boldsymbol{\xi} \in \partial G^0; \\ b_i^+ &\to 0, \quad |\mathbf{x}| \to \infty, \end{aligned} \tag{7.6}$$

with the material coefficients c_{ijkl}^{u0} as given in (3.14).

The derivations of δL and J are analogous to those in Section 3. We merely have to apply to $(3.43)^{2,3}$ the substitutions

$$\mu_0 H_i^{0+} = B_i^{0+}, \quad \mu_0 T_{ij}^{M0+} = B_i^{0+} B_j^{0+} - \tfrac{1}{2} B_k^{0+} B_k^{0+} \delta_{ij}, \tag{7.7}$$

and to put equal to zero the fields

B_0, $\mathbf{A}^{0-}$, $\mathbf{a}^-$, $\mathbf{H}^{0-}$, $\mathbf{h}^-$, $\mathbf{M}^{0-}$, $\mathbf{m}^-$, T^{M0-},

and the material coefficients

c_{ijk}^{um0}, c_{ij}^{m0}.

Thus, we deduce

$$\delta L = \int_{G^{0-}} T_{ij,j}^0 u_i \, \mathrm{d}V^0 - \int_{\partial G^0} \left(T_{ij}^0 N_j^0 + \frac{1}{2\mu_0} B_j^{0+} B_j^{0+} N_i^0 \right) u_i \, \mathrm{d}S^0 - \frac{1}{\mu_0} \int_{G^{0+}} e_{ijk} B_{k,j}^{0+} a_i^+ \, \mathrm{d}V^0, \tag{7.8}$$

where we have also used that the tangential derivative of $\mathbf{A}^{0+}$ along ∂G^0 is zero, or

$$e_{jkl} A_{i,k}^{0+} N_l^0 = 0, \quad \boldsymbol{\xi} \in \partial G^0, \tag{7.9}$$

while J becomes

$$J = -\frac{1}{2}\int_{G^{0-}} \varrho^0 c^{\mu 0}_{ijkl} u_{i,k} u_{j,l}\, dV^0 - \frac{1}{2\mu_0}\int_{G^{0+}} b_i^+ b_i^+\, dV^0$$

$$+ \frac{1}{\mu_0}\int_{\partial G^0} [\tfrac{1}{2} B_k^{0+} B_{k,j}^{0+} u_i u_j - \tfrac{1}{2} e_{ijm} B_m^{0+} A_{j,kl}^{0+} u_k u_l$$

$$+ B_k^{0+}(e_{ijk} u_l - e_{ljk} u_i)(A_{j,m}^{0+} u_m)_{,l} + \tfrac{1}{4} B_k^{0+} B_k^{0+}(u_{j,j} u_i - u_{i,j} u_j)] N_i^0\, dS^0, \tag{7.10}$$

in the derivation of which we have used

$$e_{ljk} a_{j,l}^+ N_i^0 = -e_{ljk}(A_{j,m}^{0+} u_m)_{,l} N_i^0, \quad \xi \in \partial G^0, \tag{7.11}$$

as follows from $(7.6)^5$.

The requirement $\delta L = 0$ under the constraints (7.3) yield the following set of equations and boundary conditions for the intermediate fields

$$\begin{aligned}
&B_i^{0+} = e_{ijk} A_{k,j}^{0+}, \quad e_{ijk} B_{k,j}^{0+} = 0, \quad \mathbf{x} \in G^{0+};\\
&T_{ij,j}^0 = 0, \quad T_{ij}^0 = \varrho^0 \left(\frac{\partial U}{\partial F_{i\alpha}}\right)^0 F_{j\alpha}^0, \quad \xi \in G^{0-};\\
&A_i^{0+} = \text{constant (or } B_i^{0+} N_i^0 = 0), \quad T_{ij}^0 N_j^0 = -\frac{1}{2\mu_0} B_j^{0+} B_j^{0+} N_i^0, \quad \xi \in \partial G^0;\\
&B_i^{0+} \sim \mu_0 I_0 c_i(\mathbf{x}), \quad |\mathbf{x}| \to \infty.
\end{aligned} \tag{7.12}$$

For an isotropic, homogeneous, linearly elastic and non-magnetizable superconductor the internal energy density U is given by

$$U = \frac{\mathrm{E}}{2\varrho_0(1+\nu)}\left(\frac{\nu}{1-2\nu}(\mathrm{tr}\, E)^2 + \mathrm{tr}\,(E^2)\right). \tag{7.13}$$

As done in the preceding section, we shall confine outselves here to conservative fields $\mathbf{b}^+$, i.e. as in (6.9) we introduce a potential $\psi = \psi(\mathbf{x})$, such that

$$b_i^+ = \psi_{,i}^+, \quad \mathbf{x} \in G^{0+}. \tag{7.14}$$

In order that this solution is consistent with the constraints $(7.6)^{1,5,6}$, ψ has to satisfy

$$\begin{aligned}
&\Delta\psi = 0, \quad \mathbf{x} \in G^{0\pm};\\
&\frac{\partial \psi}{\partial N^0} = (B_j u_{i,j} - B_{i,j} u_j) N_i, \quad \mathbf{x} \in \partial G^0;\\
&\psi \to 0, \quad |\mathbf{x}| \to \infty.
\end{aligned} \tag{7.15}$$

With (7.14) and (7.15) the integral over G^{0+} in (7.10) can be transformed into a surface integral, as follows,

$$- \int_{G^{0+}} b_i^+ b_i^+ \, dV^0 = \int_{\partial G^0} \psi \frac{\partial \psi}{\partial N^0} \, dS = \int_{\partial G^0} \psi (B_{i,j} u_j - B_j u_{i,j}) N_i^0 \, dS^0. \tag{7.16}$$

NOTE: We note that the potential ψ is completely determined by (7.15) (this in contrast to the potential ψ in Section 6 which still was free on the boundary ∂G). Hence, if we confine ourselves to conservative $\mathbf{b}^+$ (i.e. to (7.14)), the potential must be solved from (7.14), and there is no degree of freedom left or a determination by variation.

Confining ourselves to conservative $\mathbf{b}^+$, neglecting the influence of intermediate deformations and using the I_0-independent variables

$$\hat{\mathbf{B}} := \mathbf{B}^{0+}/\mu_0 I_0, \quad \hat{\mathbf{A}} := \mathbf{A}^{0+}/\mu_0 I_0, \quad \hat{T} := T^0/\mu_0 I_0^2, \quad \hat{\psi} := \psi/\mu_0 I_0 \tag{7.17}$$

in the buckling equation $J = 0$, we obtain analogous to the preceding section the following formula for the buckling value I_0 (omitting the hats)

$$\begin{aligned} \frac{\mathrm{E}}{\mu_0 I_0^2} = & \left\{ \int_{\partial G_0} [\psi(B_j u_{i,j} - B_{i,j} u_j) + B_k B_{k,j} u_i u_j - e_{ijm} B_m A_{j,kl} u_k u_l \right. \\ & + 2B_k (e_{ijk} u_l - e_{ljk} u_i)(A_{j,m} u_m)_{,l} + \tfrac{1}{2} B_k B_k (u_{j,j} u_i - u_{i,j} u_j)] N_i \, dS_0 \\ & \left. - \int_{G_0^-} T_{jk} u_{i,k} u_{i,j} \, dV_0 \right\} \left\{ \frac{1}{1+\nu} \int_{G_0^-} \left(\frac{\nu}{1-2\nu} e_{kk} e_{ll} + e_{kl} e_{kl} \right) dV_0 \right\}^{-1} \end{aligned} \tag{7.18}$$

where the tensor e_{ij} is the same as defined in (6.17).

8. Discussion

In the preceding sections we have derived on the basis of a variational principle explicit expressions for the magneto-elastic buckling value for two special cases, namely for a soft ferromagnetic structure and for a superconductor. Although in our opinion these two cases are from a practical point of view also the most important cases, we note that still other applications are possible. For instance, if electrical fields do play a role, one has to supplement the Lagrangian density L by electric fields, yielding (**E** is the electric field strength and **P** the polarization)

$$\mathrm{L} = \tfrac{1}{2}\varepsilon_0(\mathbf{E}, \mathbf{E}) - \tfrac{1}{2}\mu_0(\mathbf{H}, \mathbf{H}) + \varrho(\mathbf{P}, \mathbf{E}) - \varrho U. \tag{8.1}$$

Moreover, with a few adjustments, the principle can also be applied to non-linearly magnetic or to magnetically saturated media. Further possible extensions are to systems of several bodies, to bodies with internal interfaces (singular surfaces) or to infinite, but periodically supported bodies, such as rods, beams or plates.

In this paper particularly the basic theory, resulting in the two expressions: (6.22) for $B_{0,cr}$ and (7.18) for $I_{0,cr}$, is presented. Specific applications to concrete systems will be given in a forthcoming paper. Essentially this amounts to solving the problem for ψ. In this forthcoming paper, the buckling values will be calculated for systems of two parallel rods, both for the case that the rods are soft ferromagnetic and placed in a uniform magnetic field, as well as for the case of two superconducting rods with prescribed total current.

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